2000 Free Response Assignment 12

1. $e^{-x^2} = 1 - \cos x$
$x = .942$

a. $A(\text{of } R) = \int_0^{.942} \left(e^{-x^2} - (1-\cos x)\right)dx = \boxed{.591}$

b. $V = \pi \int_0^{.942} \left(\left(e^{-x^2}\right)^2 - (1-\cos x)^2\right)dx = \boxed{1.747}$

c. $V = \int_0^{.942} \left(e^{-x^2} - (1-\cos x)\right)^2 dx = \boxed{.461}$

2. a. Runner A:
$V(t) = \frac{10}{3}t$ on the interval $[0,3]$
$V(2) = \frac{10}{3} \cdot 2 = \boxed{\frac{20}{3} \frac{m}{sec}}$

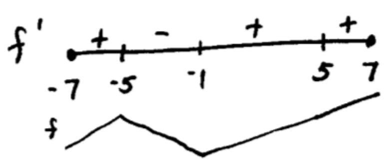
Runner B:
$V(2) = \frac{24 \cdot 2}{2 \cdot 2 + 3} = \boxed{\frac{48}{7} \frac{m}{sec}}$ or $6.857 \frac{m}{sec}$

b. Runner A:
$a(t) = v'(t) =$ slope from graph
$a(2) = \boxed{\frac{10}{3} \frac{m}{sec^2}} = 3.333 \frac{m}{sec^2}$

Runner B:
$a(t) = v'(t) = \frac{(2t+3)24 - 24t \cdot 2}{(2t+3)^2}$
$a(2) = \boxed{\frac{(4+3)24 - 24 \cdot 2 \cdot 2}{(4+3)^2} \frac{m}{sec^2}}$
$= \frac{72}{49} = 1.469 \frac{m}{sec^2}$

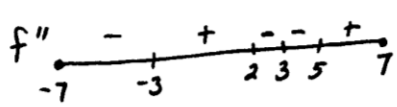
c. Runner A:
TD = area under vel. graph on $[0,10]$
$TD = \frac{1}{2} \cdot 10\,(7+10) \leftarrow$ area of trapezoid
$= \boxed{85\ m}$

Runner B:
$TD = \int_0^{10} |v(t)|\,dt$
$= \int_0^{10} \left|\frac{24t}{2t+3}\right| dt$
$= \boxed{83.336\ m}$

3. a.


f has a $\boxed{\text{rel. min. at } x = -1}$ because f' switches from negative to positive.

b. f has a $\boxed{\text{rel. max. at } x = -5}$ because f' switches from positive to negative.

c.


$f''(x) < 0$ on the intervals $\boxed{(-7,-3)\ \ (2,3) \text{ and } (3,5)}$ because f' has a negative slope.

d. There are two candidates for abs. max. ($x = -5$ and $x = 7$)
$f(7) > f(-5)$ since the area accumulated below the x-axis from -5 to -1 is much less than the area above the x-axis from -1 to 7. $\boxed{\text{The abs. max of f is at } x = 7}$

2000 Free Resp. (cont.)

4\. a. $\frac{dV}{dt} = -\sqrt{t+1}$ (neg. because it is leaking)

$dV = -(t+1)^{\frac{1}{2}}dt$

$V(\text{lost}) = \int_0^3 -(t+1)^{\frac{1}{2}}dt = -\frac{2}{3}(t+1)^{\frac{3}{2}}\Big|_0^3 = \boxed{\left(-\frac{2}{3}\cdot 4^{\frac{3}{2}}\right) - \left(-\frac{2}{3}\cdot 1\right)} = -\frac{14}{3} = -4.667 \text{ gal}$

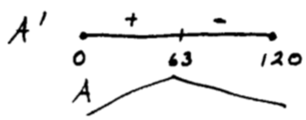
b. $V(\text{in tank at } t=3) = \boxed{30 + 8\cdot 3 - \frac{14}{3}} = 49.333 \text{ gal}$

c. $A(t) = 30 + 8t - \int_0^t (t+1)^{\frac{1}{2}}dt$

d. $A'(t) = 8 - (t+1)^{\frac{1}{2}}$

$8 - (t+1)^{\frac{1}{2}} = 0$

$t = 63$



max. at $\boxed{t = 63 \text{ min.}}$ because A' is pos. on $(0,63)$ and neg. on $(63,120)$

5\. a. $xy^2 - x^3y = 6$

$x\cdot 2y\frac{dy}{dx} + y^2 - x^3\frac{dy}{dx} - 3x^2y = 0$

$2xy\frac{dy}{dx} - x^3\frac{dy}{dx} = 3x^2y - y^2$

$\frac{dy}{dx}(2xy - x^3) = 3x^2y - y^2$

$\boxed{\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}}$

b. when $x = 1$

$y^2 - y = 6$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3, -2$

$\boxed{\text{points } (1,3), (1,-2)}$

$y'(1,3) = \frac{3\cdot 1\cdot 3 - 3^2}{2\cdot 1\cdot 3 - 1} = 0$

$y'(1,-2) = \frac{3\cdot 1(-2) - 4}{2\cdot 1(-2) - 1} = 2$

Tan. Lines

$\boxed{\begin{aligned} y-3 &= 0(x-1) \\ y+2 &= 2(x-1) \end{aligned}}$

c. vert. tan. → undef. deriv.

$2xy - x^2 = 0$

$x(2y - x^2) = 0$

$x = 0, \quad 2y = x^2$

$y = \frac{x^2}{2}$

plug in to original equation

when $x = 0$ $\quad 0 - 0 \neq 6$ there is no point

when $y = \frac{x^2}{2} \rightarrow x\left(\frac{x^2}{2}\right)^2 - x^3\left(\frac{x^2}{2}\right) = 6$

$\frac{x^5}{4} - \frac{x^5}{2} = 6$

$x^5 - 2x^5 = 24$

$-x^5 = 24$

$x^5 = -24$

$\boxed{x = \sqrt[5]{-24}}$

6\. a. $\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$

$e^{2y}dy = 3x^2dx$

$\int e^{2y}dy = \int 3x^2dx$

$\frac{1}{2}e^{2y} = x^3 + C_1$

$e^{2y} = 2x^3 + C_2$

$\ln e^{2y} = \ln(2x^3 + C_2)$

$2y = \ln(2x^3 + C_2)$

$y = \frac{1}{2}\ln(2x^3 + C_2)$

since $f(0) = \frac{1}{2} \rightarrow \frac{1}{2} = \frac{1}{2}\ln(0 + C_2)$

$1 = \ln C_2$

$e = C_2 \longrightarrow \boxed{y = \frac{1}{2}\ln(2x^3 + e)}$

b. Domain:

$2x^3 + e > 0$

$x^3 > -\frac{e}{2}$

$\boxed{x > \sqrt[3]{-\frac{e}{2}}}$

Range:

$\boxed{\text{all reals}}$